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BRIEF COMMUNICATION

SPATIALLY GROWING THREE-DIMENSIONAL WAVES ON FALLING FILM FLOW

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An important mechanism affecting the transition between the annular and mist or froth flow regimes in multiphase flow is the shearing action of the gas flow on three-dimensional waves which arise from the instability of the annular film flow. It is the propensity of film flow to be unstable with respect to three-dimensional disturbances which is of interest here. We will restrict our attention to the linear stability problem for planar film flow of a viscous liquid having nonzero surface tension in contact with an inviscid ambient phase.

The limit of linear stability for this flow can be determined solely by considering two-dimensional waves. This is a consequence of Squire's theorem which states that if a flow is stable until the Reynolds number exceeds a certain critical value, the neutral wave which marks the limit of stability at the critical Reynolds number must be two-dimensional. Squire's theorem, however, does not imply that the first manifestation of instability must be a two-dimensional wave. The most prominent wave which is frequently observed, corresponds to the most highly amplified or most unstable disturbance. Hence, it is important to determine whether the most highly amplified mode is two- or three-dimensional.

Benjamin (1961) employed Squire's transformation to infer the properties of temporally growing three-dimensional waves on planar film flow, and found that the most unstable long wave was two-dimensional. Recall that Squire's transformation shows that the stability of an oblique temporally growing disturbance depends solely on the component of the basic flow velocity profile in the direction normal to the wave front. Thus, the eigenvalues of a temporally growing three-dimensional disturbance can be related to those of a temporally growing two-dimensional disturbance at a lower Reynolds number.

It is of interest to note that most experimental studies of falling film flow have involved spatially growing rather than temporally growing waves. Shuler & Krantz (1976) recently have found that the predictions of linear stability theory for spatially growing two-dimensional disturbances agree more favorably with experiment than do the predictions based on temporally growing disturbances. These results suggest that although temporally growing two-dimensional long waves are more highly amplified than temporally growing three-dimensional waves at the same values of the flow parameters, the possibility exists that spatially growing three-dimensional waves may be more highly amplified than spatially growing two-dimensional waves. If the latter were shown to be possible, it would provide some explanation for the surprisingly early appearance of three-dimensional waves on film flows. However, the properties of spatially growing three-dimensional waves can in no way be inferred from the results presented in Shuler & Krantz (1976). This stems from the fact that Squire's transformation, when applied to spatially growing modes, indicates that three-dimensional distrubances are not equivalent to any physically significant two-dimensional disturbances. Thus, in order to determine the most unstable mode for spatially growing distrubances in viscous flows it is necessary to solve the stability problem for both two- and three-dimensional disturbances.

Relatively few studies of spatially growing three-dimensional disturbances have appeared in the literature. Gaster (1970) developed a general proof that the most highly amplified mode in an incompressible plane parallel flow of an inviscid fluid is two-dimensional for either temporally or spatially growing disturbances. However, no such general proof exists for either temporally or spatially growing disturbances in viscous flows. Furthermore, it appears that no one has attempted to determine the most unstable spatially growing mode in a viscous flow. For this reason this note determines the eigenvalues for all spatially growing long waves in falling film flow. This flow is a convenient example of a viscous flow which permits one to examine the broader question of the influence of viscosity on the characteristics of spatially growing three-dimensional disturbances.

FORMULATION OF THE LINEAR STABILITY PROBLEM

Consider a film of thickness H flowing down a plane inclined at an angle θ to the horizontal. The velocity profile of the unperturbed or basic film flow is given by $U(y) = (3/2)\bar{u}(1 - y^2/H^2)$, where $\bar{u}$ is the average velocity. The liquid has density ρ, kinematic viscosity ν, and surface tension T; in this study it will be assumed to be in contact with an inviscid gas. A Cartesian coordinate system is placed such that the x-axis is in the streamwise direction; the y-axis is perpendicular to the plane of the basic flow such that $y = 0$ is at the unperturbed gas-liquid interface; and the z-axis is in the spanwise direction.

The development of the equations of motion and associated boundary conditions appropriate to three-dimensional distrubances in this flow follows the formalism of classical linear stability theory.

The linearized equations of motion can be combined and expressed in terms of only one dependent variable, $\hat{v}$, the amplitude of the velocity perturbation in the y-direction. When nondimensionalized using H and $\bar{u}$ as the characteristic length and velocity scales, respectively, this equation assumes the form

$$\hat{v}^{iv} - 2(\alpha^2+\beta^2)\hat{v}'' + (\alpha^2+\beta^2)^2\,\hat{v} = i\alpha Re\left[\left(U - \left(\frac{\omega}{\alpha}\right)\right)(\hat{v}'' - (\alpha^2+\beta^2)\hat{v}) - U''\hat{v}\right] \qquad [1]$$

where the superscripts denote the order of differentiation with respect to y. The dimensionless boundary conditions appropriate to wavy film flow down an inclined plane, and the dimensionless surface kinematic condition also can be expressed in terms of one dependent variable

$$\hat{v} = 0 \text{ at } y = 1, \qquad [2]$$

$$\hat{v}' = 0 \text{ at } y = 1, \qquad [3]$$

$$\hat{v}'' + \left[(\alpha^2+\beta^2) - \frac{3}{\left(\frac{\omega}{\alpha} - 3/2\right)}\right]\hat{v} = 0 \text{ at } y = 0, \qquad [4]$$

$$\left[\frac{(\alpha^2+\beta^2)}{\alpha}[3\cot\theta + (\alpha^2+\beta^2)\,WeRe]\Big/\left(\frac{\omega}{\alpha} - 3/2\right)\right]\hat{v} - \left[\alpha Re\left(\frac{\omega}{\alpha} - 3/2\right) + i3(\alpha^2+\beta^2)\right] \times \hat{v}' + i\hat{v}''' = 0 \text{ at } y = 0, \qquad [5]$$

$$\hat{v} = -i\alpha\left(\frac{\omega}{\alpha} - 3/2\right) \text{ at } y = 0 \qquad [6]$$

where α and β are the dimensionless wave numbers in the x and z directions, respectively, and ω is the dimensionless frequency. This nondimensionalization introduces the Reynolds number $Re \equiv \bar{u}H/\nu$, and Weber number $We \equiv T/(\rho\bar{u}^2H)$, or equivalently, the surface tension group $\zeta \equiv We \cdot Re^{5/3}$. Boundary conditions [2] and [3] express the no-flow and no-slip conditions at the solid surface. Equation [4] expresses the condition of no drag force at the free surface. Boundary condition [5] represents the balance of the viscous, pressure, and surface tension normal stresses at the free surface. The kinematic surface condition is given by [6].

AN ASYMPTOTIC SOLUTION TO THE STABILITY PROBLEM

Equation [1] is similar in form to the Orr–Sommerfeld equation describing the linear stability of two-dimensional disturbances. Benjamin (1957) has solved the latter equation for temporally growing two-dimensional disturbances on film flow down a plane, via a power series in y. The rate of convergence of this power series will be rapid only for small Reynolds numbers such as those associated with the long waves observed in the film flow of interest here. Since the linear stability problem defined by [1] through [6] differs from that of Benjamin only in the definitions of the parameters, the solution to the present problem can be obtained by the appropriate parametric substitutions in the solution given by [4.11] in Benjamin (1957), and is given by

$$
\begin{aligned}
& i[2/3(\alpha^2+\beta^2)^2 WeRe + (\alpha^2+\beta^2)2\cot\theta] + 2\omega - 6\alpha + i\,3/2Re\left[-\frac{8}{15}\omega^2 + \frac{136}{105}\omega\alpha - \frac{24}{35}\alpha^2\right] \\
& \quad + (\alpha^2+\beta^2)(3.6\omega - 4.8\alpha) + (-9/4Re^2) \\
& \quad \times [-0.0016927\omega^3 + 0.0095945\omega^2\alpha - 0.0169753\omega\alpha^2 + 0.0102907\alpha^3] + (i\,3/2Re)(\alpha^2+\beta^2) \\
& \quad \times [-0.0723810\omega^2 + 0.2020600\omega\alpha - 0.1397615\alpha^2] \\
& \quad + (\alpha^2+\beta^2)^2(-0.0914284\omega + 0.0742865\alpha) + \left(-i\frac{27}{8}Re^3\right) \\
& \quad \times [-0.0000502\omega^4 + 0.0003382\omega^3\alpha - 0.0008003\omega^2\alpha^2 + 0.0008276\omega\alpha^3 - 0.0003175\alpha^4] \\
& \quad + (-9/4Re^2)(\alpha^2+\beta^2)[0.0021918\omega^3 - 0.0106906\omega^2\alpha + 0.017370\omega\alpha^2 - 0.0092402\alpha^3] \\
& \quad + (i\,3/2Re)(\alpha^2+\beta^2)^2[-0.0225191\omega^2 + 0.0644643\omega\alpha - 0.0468239\alpha^2] \\
& \quad + (\alpha^2+\beta^2)^3[0.0364873\omega - 0.0594137\alpha] = 0.
\end{aligned}
\qquad [7]
$$

In arriving at [7] the power series was arbitrarily truncated at sixteen terms appropriate to retaining terms of order $(\alpha Re)^3$ and α^6. Furthermore, no assumptions were made concerning whether α, β, and ω were real or complex. Thus, [7] includes both two- and three-dimensional, temporally and spatially growing modes.

For the spatially growing three-dimensional waves of interest here, $\alpha \equiv \alpha_r + i\alpha_i$ is complex, $\beta \equiv \beta_r$ is real, and $\omega \equiv \omega_r$ is real. This then constitutes an eigenvalue problem of the form

$$F(\alpha_i, \alpha_r, \beta_r, \omega_r, We, Re, \theta) = 0. \qquad [8]$$

If the parameters We, Re, and θ are specified, the wave properties α_r and α_i of infinitesimal disturbances having spanwise wave number β_r and frequency ω_r can be determined from the complex roots or eigenvalues of [7].

MOST UNSTABLE SPATIALLY GROWING MODE

A numerical scheme for determining the most highly amplified wave predicted by [7] has been developed by Shuler (1974). The spatial amplification factor $-\alpha_i$ was maximized with respect to α_r, β_r, and ω_r, for a wide range of the parameters θ, Re, and We. In all cases, when convergence was achieved, the spanwise wave number β_r was found to be zero within the single precision accuracy of the computer. This numerical search suggests that the most unstable wave is two-dimensional in the range of validity of [7]. However, a numerical search is unsatisfactory both because it cannot find a precisely zero value of β_r due to round-off errors, and because it is not possible to search all possible combinations of the parameters. Clearly an analytical proof concerning the most unstable mode is desirable.

It does not appear possible to prove analytically that the most unstable mode described by [7] is two-dimensional. However, this proof is possible if the higher-order terms in [7] are discarded, thus obtaining an approximate solution for very long waves. If [7] is truncated to terms of order α the following is obtained:

$$i[2/3(\alpha^2+\beta_r^2)^2 WeRe + (\alpha^2+\beta_r^2)2\cot\theta] + 2\omega_r - 6\alpha = 0. \qquad [9]$$

In arriving at [9] the ordering arguments $We \cdot Re = 0(1/\alpha_r^2)$ and $\omega_r = 0(\alpha_r)$ were invoked in addition to the ordering arguments implicit in the development of [7]. The real part of [9] can be solved for the frequency ω_r. The imaginary part of [9] can be solved implicitly for $-\alpha_i$ to obtain

$$-\alpha_i = -\frac{1}{6}\left\{\frac{2}{3}[\alpha_r^4 - 6\alpha_r^2\alpha_i^2 + 2\alpha_r^2\beta_r^2 + \alpha_r^4 - 2\alpha_i^2\beta_r^2 + \beta_r^4]\, WeRe + [\alpha_r^2 - \alpha_i^2 + \beta_r^2]2\cot\theta\right\}. \qquad [10]$$

The following conditions determine the most highly amplified disturbance:

$$\partial(-\alpha_i)/\partial\alpha_r = \frac{-\alpha_r}{6}\left\{\frac{8}{3}[\alpha_r^2 - 3\alpha_i^2 + \beta_r^2]\, WeRe + 4\cot\theta\right\} = 0, \qquad [11]$$

$$\partial(-\alpha_i)/\partial\beta_r = \frac{-\beta_r}{6}\left\{\frac{8}{3}[\alpha_r^2 - \alpha_i^2 + \beta_r^2]\, WeRe + 4\cot\theta\right\} = 0. \qquad [12]$$

Clearly [11] and [12] imply that either $\alpha_r = 0$ or $\beta_r = 0$ for the most unstable mode. However, when the condition $\alpha_r = 0$ is substituted into [10] and [12], all such modes are found to be stable for $\beta_r \neq 0$ and neutrally stable for $\beta_r = 0$. Hence $\beta_r = 0$, and the most unstable spatially growing long wave is two-dimensional.

A slightly more general proof can be obtained by retaining terms of order α^2 in [7] while invoking the same ordering arguments as were used in the preceding proof. However, this proof will not be given here as it is quite lengthy and adds relatively little to the conclusions of the preceding development for very long waves. The interested reader is referred to Shuler (1974).

COMPARISON BETWEEN SPATIALLY AND TEMPORALLY GROWING MODES

It is of interest to compare the predictions of the spatial and temporal formulations of this linear stability problem for three-dimensional disturbances. For temporally growing three-dimensional waves $\alpha \equiv \alpha_r$ is real, $\beta \equiv \beta_r$ is real, and $\omega \equiv \omega_r + i\omega_i$ is complex. The general solution then constitutes an eigenvalue problem of the form

$$F(\alpha_r, \omega, \beta_r, Re, We, \theta) = 0. \qquad [13]$$

The resulting temporal amplification factor ω_i was converted to an "equivalent" spatial amplification factor $-\alpha_i$ using the transformation of Gaster (1962).

$$-\alpha_i = \frac{\omega_i}{\partial\omega_r/\partial\alpha_r}. \qquad [14]$$

Figure 1 shows the spanwise wave number β_r as a function of the streamwise wave number α_r for $Re = 1.29$, $\zeta = 2.89$, and $\theta = 90°$. This value of the surface tension group, corresponding to a light mineral oil, was chosen to illustrate the properties of a highly unstable film flow at a low Reynolds number. Note that increasing values of the surface tension group, or equivalently

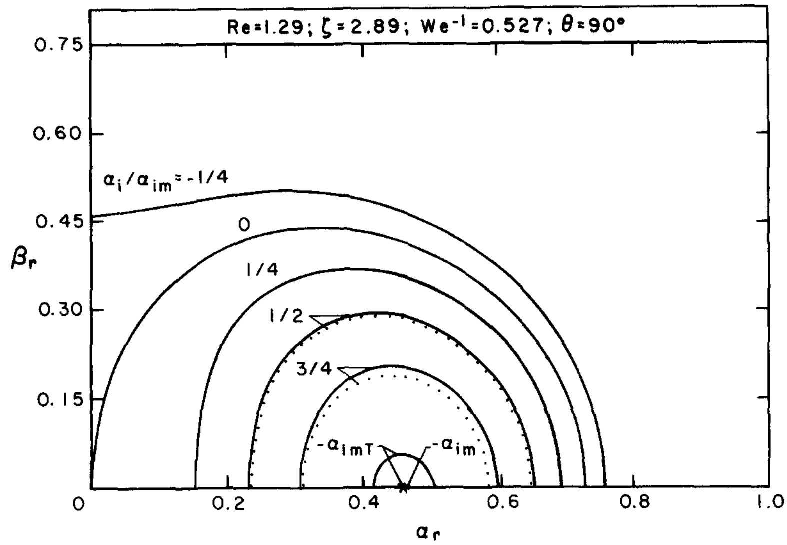

Figure 1. Spanwise wave number as a function of streamwise wave number showing contours of constant reduced spatial amplification rate predicted by the spatial formulation ———, and temporal formulation . . . . . .; $Re = 1.20$; $\zeta = 2.89$; and $\theta = 90°$.

the Weber number, have a stabilizing influence on this flow. The differences between the predictions of the spatial and temporal formulation will be more pronounced at small values of the surface tension group since the transformation of Gaster is valid only for weakly amplified disturbances. The solid contours in figure 1 correspond to fixed values of the spatial amplification factor predicted by the spatial formulation reduced with respect to α_{im}, the spatial amplification factor of the most highly amplified wave predicted by the spatial formulation. The dotted contours correspond to fixed values of the spatial amplification factor predicted by the temporal formulation reduced with respect to α_{im}. For values of $\alpha_i/\alpha_{im} < 0.5$ the predictions of the two formulations were nearly identical. The most highly amplified wave predicted by both formulations corresponds to $\beta_r = 0$, thus implying two-dimensional disturbances. The temporal formulation predicts a maximum amplification factor $\alpha_{imT} = 0.0430$ at $\alpha_r = 0.461$. However, this value of $-\alpha_i$ corresponds to a band of waves in the spatial formulation indicated by the solid contour for $-\alpha_{imT}$; that is, the spatial formulation predicts a higher amplification rate for the most unstable wave given by $\alpha_{im} = 0.0440$ at $\alpha_r = 0.469$. Since our results suggest that the spatial amplification rate of the most unstable wave is increasing with Reynolds number, the difference between the predictions of the spatial and temporal formulations might be expected to be more pronounced at higher Reynolds numbers. However, it is doubtful[7] can be applied at Reynolds numbers significantly greater than unity.

CONCLUSIONS

This represents the first analysis of spatially growing three-dimensional disturbances in falling film flow, and appears to be the first analysis of spatially growing three-dimensional disturbances in a viscous flow of any type. A comparison between the predictions for spatially and temporally growing three-dimensional disturbances indicates that the amplification rates for disturbances in this flow can be sufficiently large, even at low Reynolds numbers, to result in a difference between the predictions of the spatial and temporal formulations of this linear stability problem. The principal conclusion of this note is that the most unstable spatially growing mode is two-dimensional for long waves.

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